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Structures of BCI and Open Problems

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Definition 1. Let X be a partially ordered set with a binary operation * and a constant 0. X is called a **BCI** if * satisfies the following conditions:

BCI 1) $(x*y)*(x*z)\leq z*x$,

BCI 2) $x*(x*y)\leq y$,

BCI 3) $x*x=0$,

BCI 4) $x*y=0\Leftrightarrow x\leq y$,

BCI 5) $x\leq 0\Rightarrow x=0$.

Remark 1. The following facts on BCI are well-known.

1) If X stisfies BCI 1) - 4) and

BCK 5) $0*x=0$,

then X is called **BCK**.

2) A.Ursini introduced a concept(1994) which is called a **subtractive algebra**(see A.Ursini). This is a generalization of BCK, BCI. The axioms are given by the following two (very simple) identities:

$$x*0=x, \qquad x*x=0.$$

3) It is well-knwon that the classes of BCK, BCI make (proper) quasivarieties(A.Wronski and Y.Komori(1983)). The quasivariety of BCI has the relative congruence extension property(J.G.Raftery-C.J.van Alten(1998)).

4) BCI is not algebraizable(W.Blok-D.Pigozzi(1989)).

5) BCI is exactly the residuation subreducts of sir comonoids(J.G.Raftery-C.J. van Alten(1998)).

6) A BCI in which

$$x * (x * y) = y * (y * x)$$

holds for any x, y is a BCK.

7) A BCI in which

$$(x * y) * y = x * y$$

holds for any x, y is a BCK.

8) A BCI in which

$$x * (y * x) = x$$

holds for any x, y is a BCK.

9) There is a method of making a (new) BCI from BCK and BCI(Xin Li(1984)). But we do not find all new BCI by this way.

On the other hand, the following basic properties hold in any BCI:

(1) $x \leq y$ and $y \leq x \Rightarrow x = y$.

(2) $x \leq y \Rightarrow z * y \leq z * x, x * z \leq y * z$.

(3) $x * 0 = x$.

(4) $(x * y) * z = (x * z) * y$.

(4) is called the **Permutation Rule**. The terminology is due to A.Avron of Tel Aviv.

Definition 2. $B = \{x \mid x \geq 0\}$ is called the **BCK-part** of X.

Definition 3. If there exists an element a such that $a \in X - B$, the set consisting of all elements which are comparable with a plus all elements which associate with $a(a, b$ associate $\Leftrightarrow \exists c(c \leq a, b))$ is called a **branch of X.** The branch including a is denoted by $B(a)$.

Repeating this procedure, we have a decomposition of X:

$$X = B(0) \cup B(a) \cup B(b) \cup \ldots$$

They are the very important concepts to research BCI.

From now $x * y$ is denoted by multiplicatively xy.

Proposition 1.

(1) $B(0)$ is a subalgebra of X.

(2) $B(0)$ is the maximal BCK ideal of X.

(3) $x \in B(0), y \in X - B(0) \Rightarrow xy, yx \in X - B(0)$.

(4) $y(y(yx)) = yx$. As a special case, $(0(0(0x))) = 0x$.

(5) $(0x)(0y) = 0(xy)$.

Proof. (1) A proof of "$x, y \in B \Rightarrow xy \in B$." $0 \leq x$ implies $0y \leq xy$. From BCI 4), $0y = 0$, So $xy \in B$.

(2) Let $a \in X - B$. For some $x \in B$, we assume $xa \in B$. Then $x(xa) \in B$ and $0 \leq x(xa) \leq a$. Hence $a \in B$, which is a contradicion. This implies

$$x \in B, a \in X - B \Rightarrow xa \in X - B.$$

Moreover, $ax \in X - B$ holds.

$$(ax)a = (ax)(a0) \leq 0x = 0.$$

From BCI 5), $(ax)a = 0$, which implies $ax \leq a$. If $ax \in B$, then $a \in B$, which is impossible. So $ax \in X - B$. This shows that B is a maximal BCK subalgebra of X.

To prove that B is an ideal of X, we assume $xy, y \in B$. If $x \in X - B$ then $xy \in X - B$. Hence B is an ideal of X.

(3) From (2), this is trivial.

(4) From BCI 2),

$$y(y(yx)) \leq yx.$$

On the other hand, BCI 1) and the permutation rule imply

$$(yx)(y(y(yx))) \leq (y(yx))x = (yx)(yx) = 0.$$

(5) We mention very simple and interesting proof given by C.Xi.

$$((0x)(0y))(0(xy)) = ((0(0y))x)(0(xy)) = (0(0y))(0(xy))x$$
$$\leq ((xy)(0y))x \leq (x0)x = (xx)0 = 00 = 0.$$

So we have $(0x)(0y) \leq 0(xy)$. Conversely,

$$0(xy) = ((0y)(0y))(xy) = ((0y)(xy))(0y) \leq (0x)(0y).$$

Therefore, we have the identity: $(0x)(0y) = 0(xy)$.

Proposition 2.

(1) If x is the samllest element of a branch, $a(ax) = x$. In particular, $0(0x) = x$.

(2) If $0(0x) = x$ then x is the smallest element in the branch $B(x)$.

(3) If $x \notin B(0)$ then $0x$ is the smallest element of some proper branch.

(4) The smallest element of B(x) is given by $0(0x)$. Therefore each branch has the smallest element.

Proof. (1) BCI 2) $\Rightarrow a(ax) \leq x$. Since x is the smallest element,

$$a(ax) = x$$

is obtained. As a special case, we have $0(0x) = x$.

(2) Assume that $0(0x) = x$. Let $y \leq x$. Then $yx = 0$. On the other hand,

$$xy = (0(0x)y = (0y)(0x) \leq xy = 0.$$

So we have $x = y$.

(3) From Proposition 1(4),$0(0(0x)) = 0x$. Hence $0x$ is the smallest element of the branch with $0x$. Morover, $x \notin B(0)$ implies $0x \notin B(0)$. The branch is proper.

(4) $0 = (0x)(0x) = (0(0x))x \to 0(0x) \leq x$.

Hence $0(0x)$ is the smallest element, and this belongs to the branch $B(x)$

From the above discussion, we have the following **Structure Theorem on BCI.**

Theorem. BCI has two main parts: BCK-part,and an Abelian group consisting of all smallest elements.

If BCK part is trivial, then all branches are trivial and BCI reduces an Abelian group.

we refer as the **base** the set(Abelian group part) consisting of all smallest elements.

To develop such an abstract theory, various examples are useful and helpful for research workers. In particular, the following examples were used. The main parts in Propositions and Theorem are contained in each example(Examples 2 and 3).

Example 1. An additively written Abelian group is BCI under $x * y = x - y$. If the BCK-part of a BCI is trivial(namely $B = \{0\}$), each branch is also trivial, and the BCI is an Abelian group.

The first example on BCI was Example 2, and they were analyzed in detail. Consequently, we obtained some elementary properties of BCI. $\{0, a\}$ is their BCK-part in both BCI. $\{0, 1\}, \{0, 1, 2\}$ are Abelian group parts respectively.

The first unsolved problem is to find an algorithm to describe all finite BCI.

Example 2.

$*$	0	a	1
0	0	0	1
a	a	0	1
1	1	1	0

*	0	a	1	2
0	0	0	2	1
a	a	0	2	1
1	1	2	0	2
2	2	1	1	0

Example 3.

*	0	1	2	3	4	5
0	0	0	3	2	3	3
1	1	0	3	2	3	3
2	2	2	0	3	0	0
3	3	3	2	0	2	2
4	4	2	1	3	0	1
5	5	2	1	3	1	0

*	0	1	2	3	4	5
0	0	0	2	2	2	2
1	1	0	2	2	2	2
2	2	2	0	0	0	0
3	3	2	1	0	1	0
4	4	2	1	1	0	0
5	5	2	1	1	1	0

*	0	1	2	3	4	5
0	0	0	3	2	3	2
1	1	0	5	4	3	2
2	2	2	0	3	0	3
3	3	3	2	0	2	0
4	4	2	1	5	0	3
5	5	3	4	1	2	0

$*$	0	1	2	3	4	5
0	0	0	2	3	4	2
1	1	0	5	3	4	2
2	2	2	0	4	3	0
3	3	3	4	0	2	4
4	4	4	3	2	0	3
5	5	2	1	4	3	0

Let us mention some unsloved problms on BCI. We do not know many facts about infinite (proper) BCI. By a proper BCI, we mean the BCK-part and the base are not trivial.

(1) Find all BCI with a given BCK as its BCK-part and a given Abelian group as its base.

(2) Give some examples of infinite proper BCI with any cardinality. We can obtain a BCI by adding one element to a given BCK.

(3) Are there BCI such that BCK-part is the cardinality $\aleph_\alpha$ and the base is the cardinality $\aleph_\beta$? We can construct BCK with any cardinality $\aleph_\alpha$.

(4) Characterize termal hyperidenties of class BCK-algebras(BCH, BCC). Characterize algebras with termal hyperidentities of BCK-algebras(BCH, BCC). (for detail, see Yu.Movsisyan (8)). The same problems are considered for BCI.

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